Probability Distribution of Spectral Hole Duration in Cognitive Networks

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***Abstract*—Operation of cognitive secondary networks is critically dependent on the activity patterns of primary users. In this paper, we investigate the probability distribution of spectral holes, assuming that active and idle periods of primary users are independent random variables (which need not be identically distributed). We consider black, white, and gray holes, which correspond to time intervals when all channels are busy, all channels are idle, and some channels are busy while others are idle, respectively. We show that the duration of black and white holes may be described using an exponential approximation which holds regardless of the actual probability distribution of channel active and idle times, as long as the number of channels is not too small. The time interval between successive black hole occurrences is shown to be exponentially distributed as well. We also analyze the behavior of gray holes and quantify their impact using a simple proxy measure.**

I. Introduction

Opportunistic spectrum access (OSA), also called cognitive or dynamic spectrum access, allows the channels in the working band to be shared between licensed (primary) and unlicensed (secondary) users. To ensure priority access for primary users, unlicensed users may use a channel as long as none of the licensed users uses that same channel; as soon as a primary user begins its activity on a channel, secondary users must move to another channel. In this manner, bandwidth utilization may be improved without compromising the spectrum usage rights of primary users [4]. OSA technology is used in local and wide-range networks [3], [8], personal area networks [6] and, more recently, cellular networks such as unlicensed LTE [2], [19].

Timely detection of both beginning and end of primary user activity is accomplished through spectrum sensing [13] which may be performed by one or more nodes in an OSA network. Channel state information obtained by sensing is used to decide on the selection of the working channel (or channels) in the next epoch. The epoch may last until the beginning of primary user activity on the new channel, or it may have a predefined duration, fixed or adjustable, essentially equivalent to frequency hopping [10]. Frequency hopping in OSA networks may follow a simple linear sequence [9], a predefined pseudorandom sequence similar to the one used in IEEE 802.15.1 medium data rate wireless personal area (Bluetooth) networks [12], or an adaptive pseudorandom sequence dynamically adjusted to the channel occupancy patterns [14], [18]. As the operation of an OSA network is critically dependent on the availability of channels free from primary user activity, a study of spectrum availability is a prerequisite for studies of many important aspects of OSA network operation, including spectrum sensing, spectrum analysis, and spectrum decision [1].

When discussing spectrum availability, three distinct cases may be observed. Time intervals during which all channels in the working band are used by primary users will be referred to as ‘black holes’ or total blackouts, while the term ‘white holes’ will denote the time intervals where all channels are free from primary user activity and available to OSA network or networks. However, most of the time some of the channels in the working band will be occupied by primary users while others will be idle and, consequently, available for secondary network use; these time intervals will be referred to as ‘gray holes’ or partial blackouts.

In this paper, we extend our preliminary analysis from [15] to examine the behavior of black, gray, and white spectral holes and analyze the probability distribution of their duration. We assume that active and idle periods of primary users are independent random variables which need not be identically distributed, and calculate mean value, coefficient of variation, and skewness for the duration of spectral holes. We then derive an approximation for the duration of the black hole based on an exponential distribution; the approximation holds when the number of channels is high enough, regardless of the initial distribution of active and idle times on the channels. We validate the approximation using exponential, Erlang-2, and mixed distribution with constant active and exponentially distributed idle times, as well as a combination of these distributions. We also show that the time interval between successive black hole occurrences is nearly exponentially distributed. Finally, we analyze the behavior of white holes, and quantify the impact of gray holes using a simple proxy measure.

The paper is organized as follows. In Section II we use probabilistic analysis to model primary user activity and the conditions which lead to a hole. Section III presents the derivation of a simple yet effective approximation for the probability distribution function of blackout duration. Sections IV and V present similar analyses for white and gray holes, respectively, while Section VI concludes the work.

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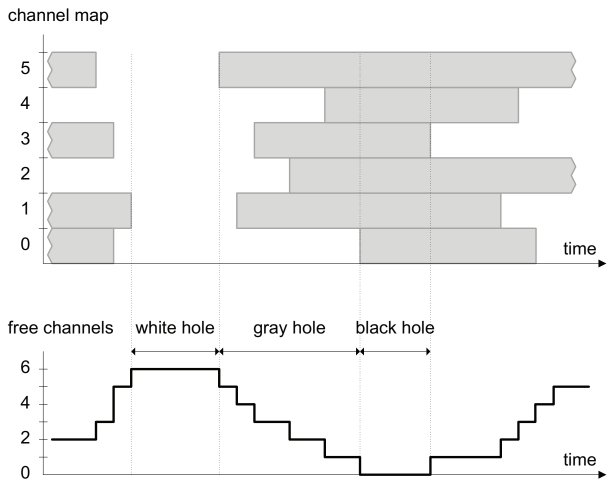


Fig. 1. Pertaining to the definition of black, gray, and white spectrum holes.

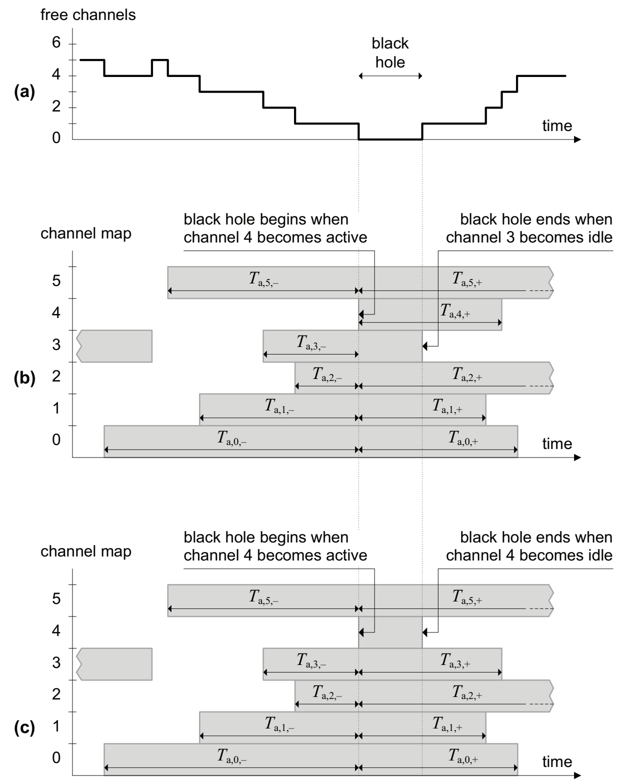


Fig. 2. A black hole (a) may begin on one channel and end on another one (b), or begin and end on the same channel on(c).

II. System model

We assume that the working band consists of N channels, each of which is intermittently used by a unique primary user. Primary users are heterogeneous with respect to probability distributions of active and idle channel times; there are K types of primary users, with users of type k occupying N_k channels (so that $\sum_{k=1}^{K} N_k = N$). Durations of active and idle times for primary users of type $k = 1\,..\,K$ are denoted as $T_{a,k}$ and $T_{i,k}$, and follow random probability distributions with the probability density functions (pdf-s) $t_{a,k}(x)$ and $t_{i,k}(x)$, respectively. As active and idle times alternate, pdf of the total cycle time on a channel of type k may be obtained as a convolution of pdf-s of the corresponding active and idle times:

$$t_k(x) = t_{i,k}(x) * t_{a,k}(x) \tag{1}$$

The probability that the channel is busy (idle) can be calculated as

$$p_{on,k} = \frac{\overline{T_{a,k}}}{\overline{T_{a,k}} + \overline{T_{i,k}}} \tag{2}$$

$$p_{\mathit{off},k} = 1 - p_{on,k} \tag{3}$$

while mean cycle time is $\overline{T_{cyc,k}} = \overline{T_{a,k}} + \overline{T_{i,k}}$.

Three types of spectral holes defined above are schematically depicted in Fig. 1, in which the top diagram represents the status of the channels in the working band (assuming the band contains $N = 6$ channels), while the bottom diagram show the corresponding number of idle channels in the working band. During a white hole the number of idle channels is $N_i = N$. A gray hole corresponds to the time interval when there are both busy and idle channels in the working band, i.e., $0 < N_i < N$. Finally, a black hole corresponds to the time interval during which there are $N_i = 0$ idle channels. Collisions between primary and secondary users occur due to black holes and sometimes due to gray holes as well. The resulting disruptions force the secondary network to recover; in some cases, it may even have to re-form anew. Detailed analysis of OSA network behavior in these cases is beyond the scope of this paper; some preliminary results can be found in [17], [16].

III. Modeling black holes

Modeling of black holes can be approached using renewal theory as the major analytical tool. Namely, a random point process $\psi = \{t_i\}$ for which the interarrival times $\{X_i\}$ are positive, independent, identically distributed random variables represents a *renewal* process [7]. The beginning of new cycle is a *renewal point* at which a new probabilistic replica of the original process starts. Furthermore, consider an arbitrary event occurring at the time t relative to the beginning of a new cycle of duration X_i. The period from renewal point to t, denoted as $X_{i,-}$, is referred to as elapsed or deficit cycle time; similarly, the period from the time t to the $X_{i,+} = X_i - t$ is referred to as residual or excess cycle time. Elapsed and residual cycle times are known to have identical probability distributions [7].

In the context of a cognitive network, cycle time is comprised of active and idle channel periods. The process that counts cycles of primary activity starting from the onset of active period on some channel is a renewal process. Renewal points correspond to the beginning of active channel periods.

A black hole begins when primary users are already active on $N-1$ channels and the last remaining idle channel turns busy. It ends when any one of the channels turns idle, as shown in Fig. 2.

Let us assume that all channels except channel 4 have already become active, and that the black hole begins by the onset of primary user activity begins on channel 4. Beginning of the black hole is a random point in the active period for each of the channels that have already switched to active state. Therefore, the time between the beginning of the active period on channel i and the beginning of the black hole is elapsed active time, which we will denote with $T_{a,k,-}$ (assuming that the channel is used by primary user of type k). The time between the beginning of activity on channel 4 (i.e., the black hole) and the end of active period on any other channel $i \neq 4$ is residual active channel time, denoted with $T_{a,k,+}$ in Fig. 2.

Let us denote the cumulative distribution function (CDF) of elapsed active channel time (with primary user of type k) with $B_k(x) = P(T_{a,k,-} \leq x)$, while its pdf is $b_k(x) = \frac{dB_k(x)}{dx}$. Let us also define $P(T_{a,k,-} > x) = 1 - B_k(x) = T^c_{a,k}(x) = \int_{y=x}^{\infty} t_{a,k}(y)dy$.

Then, the CDF $B_k(x)$ can be calculated as

$$B_k(x) = \frac{1}{\overline{T_{a,k}}} \int_0^x T^c_{a,k}(y)dy \tag{4}$$

and the pdf of deficit channel active time as

$$b_k(x) = \frac{d}{dx} B_k(x) = \frac{T^c_{a,k}(x)}{\overline{T_{a,k}}} \tag{5}$$

By the same token, elapsed and residual idle times for primary user of type k, have probability distributions described by

$$A_k(x) = \frac{1}{\overline{T_{i,k}}} \int_0^x T^c_{i,k}(y)dy \tag{6}$$

$$a_k(x) = \frac{d}{dx} A_k(x) = \frac{T^c_{i,k}(x)}{\overline{T_{i,k}}} \tag{7}$$

Since $T_{a,k,+} = T_{a,k,-}$ [7], we will first calculate $T_{a,k,-}$ using the known probability density function of channel active time $t_{a,k}(y)$. Namely, $B_k(x) = P(T_{a,k,-} \leq x) = P(T_{a,k,+} \leq x)$, while $A_k(x) = P(T_{i,k,-} \leq x) = P(T_{i,k,+} \leq x)$.

If the channel that started the black hole differs from the one that ends it, as in Fig. 2 (b), the CDF of black hole duration D is

$$D_d(x) = 1 - \prod_{k=1}^{K} (1 - B_k(x))^{N_k} \tag{8}$$

However, if the same channel begins and ends the black hole, as in Fig. 2 (c), the CDF of black hole duration can be calculated as

$$D_s(x) = \sum_{k=1}^{N} \frac{N_k}{N} T_{a,k}(x) \tag{9}$$

The latter scenario occurs if the duration of active channel time on the channel which started the black hole is smaller than the duration of residual active channel time on the remaining $N-1$ channels. Using an auxiliary variable defined as

$$\begin{aligned}\Pi_{tot} &= \prod_{k=1}^{K} \left(\int_{x=0}^{\infty} P(T_{a,k} \leq x) P(T_{a,k,+} = x) \right)^{N_k} \\ &= \prod_{k=1}^{K} \left(\int_{x=0}^{\infty} \left(\int_{y=0}^{x} t_{a,k}(y)dy \right) b_k(x)dx \right)^{N_k}\end{aligned} \tag{10}$$

we can obtain the probability that the black hole begins and ends on the same channel as

$$P_{lb} = \sum_{k=1}^{N_k} \frac{\Pi_{tot}(N_k/N)}{\int_{x=0}^{\infty} \left(\int_{y=0}^{x} t_{a,k}(y)dy \right) b_k(x)dx} \tag{11}$$

For constant values of channel active time, $P_{lb} = 0$.

By combining the two scenarios, we can obtain the CDF for black hole duration as

$$D(x) = (1 - P_{lb}) D_d(x) + P_{lb} D_s(x) \tag{12}$$

and its pdf as $d(x) = \frac{dD(x)}{dx}$. The mean value and variance of black hole duration can be calculated as

$$\overline{D} = \int_{x=0}^{\infty} x d(x) dx \tag{13}$$

$$var(D) = \int_{x=0}^{\infty} (x - \overline{D})^2 d(x) dx \tag{14}$$

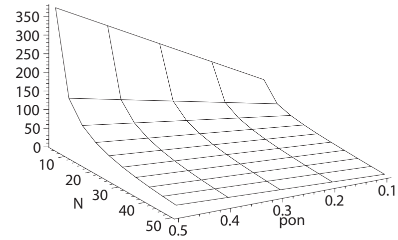
We note that expressions (8), (9) and (12) contain linear combinations of CDF's of products of channel active time and residual active time. This hints that the variability of black hole duration might be larger than the variability of the corresponding active or residual active time for a single channel. To describe this dependence in quantitative terms, it may be of interest to consider the coefficient of variation, defined as the ratio of standard deviation and mean value, $c_{v,b} = \sqrt{var(D)}/\overline{D}$, and skewness, defined as

$$S_d = \frac{(-1)^3 \left.\frac{d^3 D^*(s)}{ds^3}\right|_{s=0} - 3 \left.\frac{d^2 D^*(s)}{ds^2}\right|_{s=0} \overline{D} + 2\overline{D}^3}{var(D)^{3/2}} \tag{15}$$

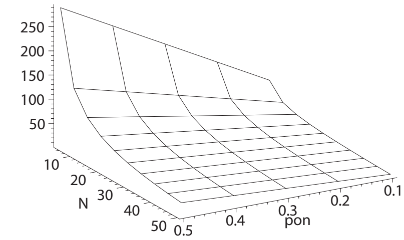
where $D^*(s)$ denotes the Laplace-Stieltjes Transform (LST) of the probability distribution for the duration of complete black hole, defined as $D^*(s) = \int_{x=0}^{\infty} e^{-sx} d(x)dx$. Skewness measures the symmetry of the distribution around the mean value; values close to zero indicate a symmetric probability distribution, while large positive or negative values indicate that the probability distribution has a long tail to the right or left of the mean, respectively.

A. Characterization of black holes

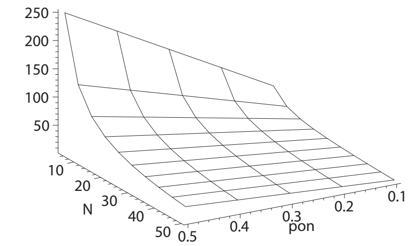
To evaluate the behavior of black holes, we have calculated their mean value, coefficient of variation, and skewness, assuming homogeneous primary sources with active and idle times following exponential, Erlang-2, and a mixed distribution with constant active and exponential idle times. Activity factor was varied between $p_{on} = 0.1$ and 0.5, while the



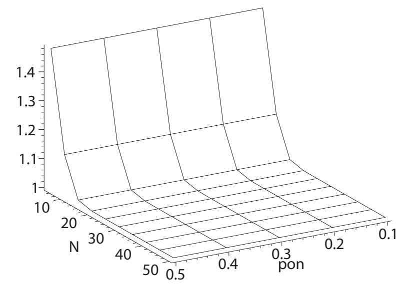
(a) Mean value, exponential distribution of active and idle times.



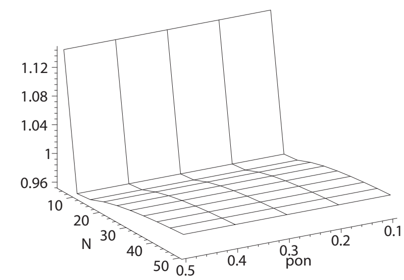
(b) Mean value, Erlang-2 distribution of active and idle times.



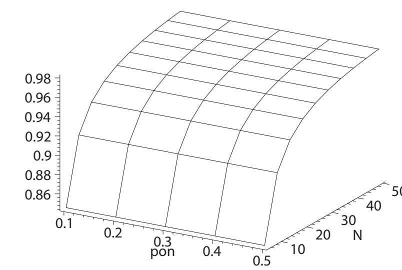
(c) Mean value, mixed distribution.



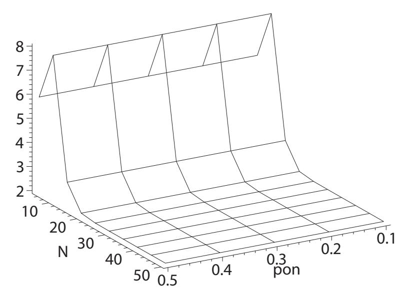
(d) Coefficient of variation, exponential distribution of active and idle times.



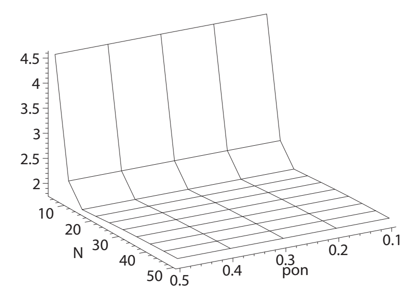
(e) Coefficient of variation, Erlang-2 distribution of active and idle times.



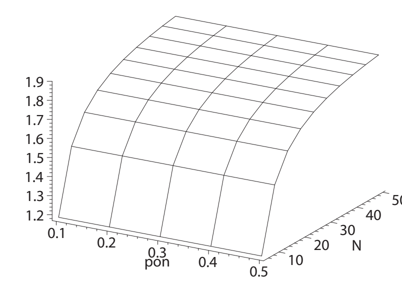
(f) Coefficient of variation, mixed distribution.



(g) Skewness, exponential distribution of active and idle times.



(h) Skewness, Erlang-2 distribution of active and idle times.



(i) Skewness, mixed distribution.

Fig. 3. Black hole duration statistics vs. number of channels and activity factor of primary user activity. Mixed distribution refers to constant active and exponentially distributed idle times.

number of channels was varied between $N = 5$ and 50. Time is expressed in normalized unit slots, to avoid over-reliance on any particular technology. Fig. 3 shows statistical descriptors of black hole duration for mean cycle time of primary source set to $T_{cyc} = 3000$ unit time slots.

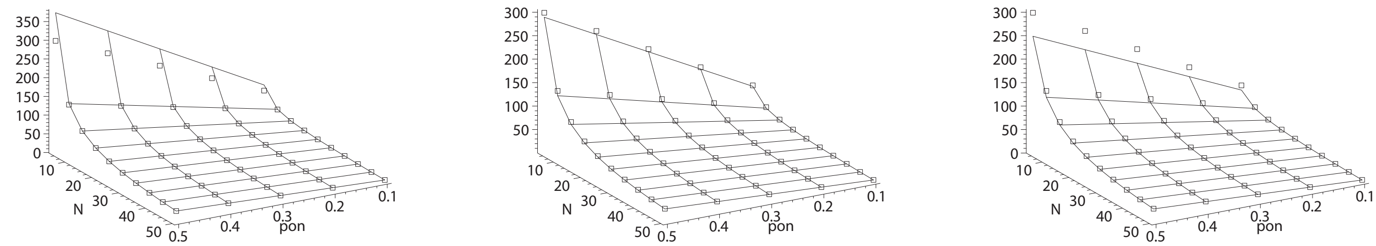
As can be seen, mean duration of a black hole decreases quickly when the number of channels increases, but increases when the primary user activity factor increases. The increase is gradual, but the slope is considerably higher at the low number of channels (5 or 10). All distributions exhibit similar behavior, although the mean duration is highest for exponential distribution and lowest for the mixed distribution.

For exponential distribution, both coefficient of variation and skewness of black hole duration are hyperexponential, reaching about 1.4 and 8, respectively, at low number of channels. Skewness values of this value point to a long tail distribution, which means that very long black holes can occur. Both coefficient of variation and skewness quickly drop to values of about one and two, respectively, when the number of channels exceeds 20 or so, and both are virtually unaffected by primary user activity factor.

Similar behavior may be observed in the diagrams obtained with Erlang-2 and mixed distribution, except that in the former case the distribution exhibits hyperexponential behavior, while in the latter is slightly hypoexponential. In all cases, both coefficient of variation and skewness reach their limiting values quickly as the number of channels increases above 15. Since those limiting values (one and two, respectively) are characteristic for exponential distribution, it may be of interest to investigate asymptotic probability distribution of black hole duration.

B. *Asymptotic distribution of black hole duration*

Let us approximate the CDF of black hole duration as $D(x) \approx D_d(x)$, as the probability P_{lb} from (11) is very small.

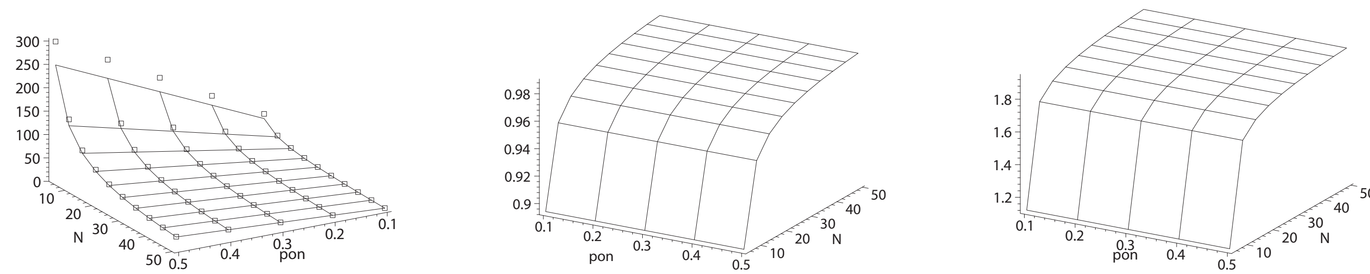


(a) Exponential distribution of active and idle times.

(b) Erlang-2 distribution of active and idle times.

(c) Mixed distribution.

Fig. 4. Mean value of black hole duration (lines) with superimposed exponential approximation (squares). Mixed distribution defined as above.



(a) Mean value.

(b) Coefficient of variation, combined distribution.

(c) Skewness, combined distribution.

Fig. 5. Black hole duration statistics vs. number of channels and activity factor of primary user activity. Channel active and idle times distributed according to a combination consisting of 40% channels with exponential, 40% channels with Erlang-2, and 20% channels with mixed distribution defined above.

Then, by applying the limit

$$\prod_{k=1}^{K}(1-B_k(x))^{N_k} = \prod_{k=1}^{K}\left(1-\frac{N_k B(x)}{N_k}\right)^{N_k} \approx e^{-\sum_{k=1}^{K} N_k B_k(x)} \tag{16}$$

we can obtain CDF of the black hole duration as

$$D(x) \approx 1 - e^{-\sum_{k=1}^{K} N_k B_k(x)} = 1 - e^{-\sum_{k=1}^{K} \frac{N_k B_k(x)}{x} x} \tag{17}$$

Since all expressions $(1-B_k(x))^{N_k}$ have a steep negative slope of $-N_k b_k(0)$ around the point $x = 0$ whenever the number of channels is high enough (say, when $N_k \geq 4$), we can approximate $B_k(x)$ by expanding it into a Taylor series at the point $x = 0$ and retaining only the first two components:

$$\frac{B_k(x)}{x} \approx \frac{B_k(0) + b_k(0)x}{x} \tag{18}$$

Since $B_k(0) = 0$ for any probability distribution, it follows that around point $x = 0$ we have $\frac{B_k(x)}{x} \approx b_k(0)$ which leads us to the final approximation

$$D(x) \approx 1 - e^{-\sum_{k=1}^{K} N_k b_k(0)x} = 1 - e^{-\lambda_d x}. \tag{19}$$

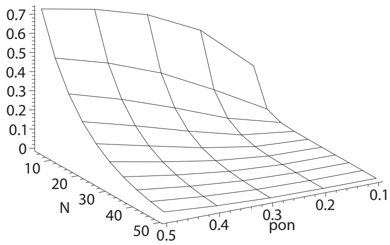
In other words, the probability distribution of black hole duration may be approximated by an exponential distribution with the rate of $\lambda_d = \sum_{k=1}^{K} N_k b_k(0)$, where $b_k(0)$ is the value of pdf of residual active time of the primary channel of type k at the point $x = 0$. The approximation becomes better (i.e., closer to the true value) as the number of channels $N = \sum_{k=1}^{K} N_k$ increases.

To assess the accuracy of the approximation, Fig. 4 shows mean value of black hole duration – essentially, the same information shown in Figs. 3(a), 3(b), and 3(c) – but with superimposed values for exponential approximation (19), shown as squares. As can be seen, beginning with $N = 10$ channels or more, the exponential approximation produces results that are virtually indistinguishable from the true value obtained from (12).

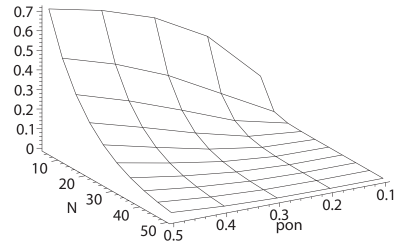
As further proof of the feasibility of our analysis in case of different distributions of active and idle times on different channels, we have considered a combination in which 40% of channels use exponential distribution of active and idle times, 40% of channels exhibit Erlang-2 distribution of these times, while the remaining 20% of channels use mixed distribution with constant active and exponentially distributed idle times. The results are shown in Fig. 5, again with superimposed exponential approximation for mean value in Fig. 5(a). As can be seen, the approximation shows very good match with the actual distribution, while the coefficient of variation and skewness are close to the values obtained with individual distributions in the combination.

C. Duration of black holes

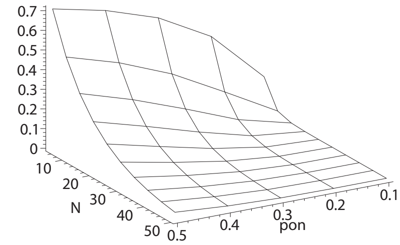
We have also calculated the probability that the duration of a black hole is longer than 100 and 200 time units, assuming (as before) that the mean value of primary activity cycle time



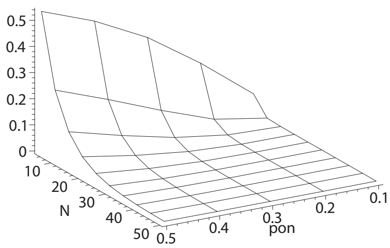
(a) Probability that a black hole is longer than 100 slots, exponential distribution of active and idle times.



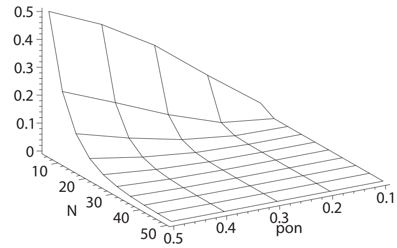
(b) Probability that a black hole is longer than 100 slots, Erlang-2 distribution of active and idle times.



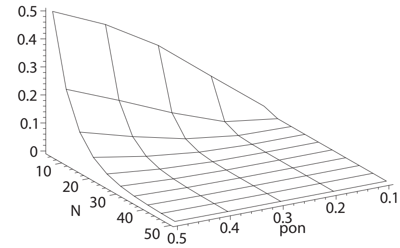
(c) Probability that a black hole is longer than 100 slots, combined distribution.



(d) Probability that a black hole is longer than 200 slots, exponential distribution of active and idle times.



(e) Probability that a black hole is longer than 200 slots, Erlang-2 distribution of active and idle times.



(f) Probability that a black hole is longer than 200 slots, combined distribution.

Fig. 6. Probability that black hole duration is longer than 100 (top row) and 200 slots (bottom row) for primary sources. Combined distribution as defined above.

is 3000 units. The results are shown in Fig. 6, for exponential, Erlang-2, and combined distribution defined above, in the left, middle, and right columns, respectively.

The following observations can be made. First, observed probability values are higher for shorter black hole duration (as expected). However, the diagrams that correspond to different distributions are rather similar, the differences being less than 10% throughout the observed range of values of independent variables. We may therefore conclude that the probability under consideration is virtually independent of the actual distribution of active and idle times on the channel. This provides additional corroboration towards the validity of the exponential approximation proposed above.

Second, observed values of probability decrease rapidly with the increase in the number of channels N. While these probabilities are higher for higher values of primary user activity factor (duty cycle) p_{on}, even this dependency diminishes when the number of channels increases. As before, we may conclude that operation with a larger number of channels is beneficial for a cognitive network since it reduces the impact of random primary user activity.

D. Inter-black hole interval

It is also of interest to find the probability distribution of the time interval between two consecutive black hole events. Probability of black hole occurrence is $P_b = \prod_{k=1}^{K}(p_{on,k})^{N_k}$. Assuming slotted time access by primary users, the probability generating function (PGF) for this time interval can be derived from the geometric distribution for the occurrence of a black hole and the pdf-s of active and idle channel times as

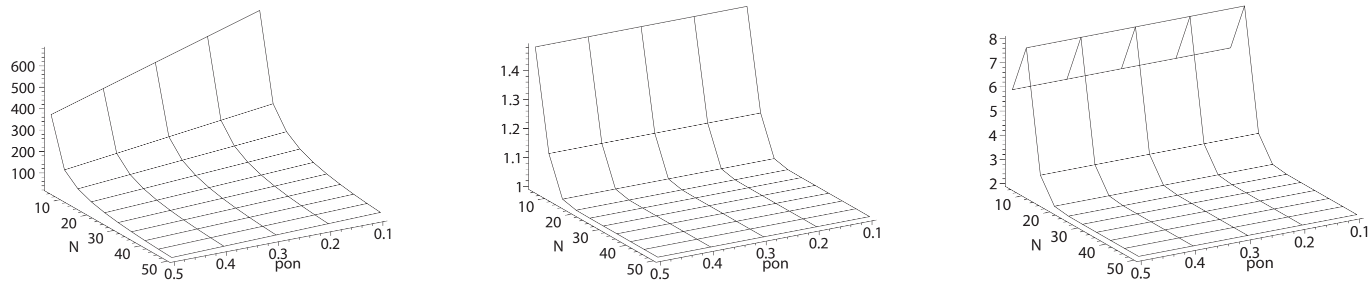
$$I_d(z) = \sum_{k=0}^{\infty} P_b(1-P_b)^k z^{k+1} = \frac{P_b z}{1-(1-P_b)z} \tag{20}$$

with mean value of $\overline{I_d} = \frac{1}{P_b}$. Alternatively, under the assumption of continuous time model of primary user access, we can model the time interval between consecutive black holes as exponentially distributed with a rate of $\lambda_i = P_b$, since the probability of zero arrivals in a small unit period is $1 - P_b$, for geometrically distributed inter-black hole interval, and approximately $1 - \lambda_i$, for exponentially distributed inter-black hole intervals.

IV. Modeling white holes

Using the same approach as above, we can calculate the duration of the time interval when all channels are idle by looking into the last channel which starts the complete availability of channels and the channel which switches to busy state and 'breaks' this complete availability. White hole duration can be described with a CDF of $W(x) = (1-P_{lw})W_d(x) + P_{lw}\sum_{k=1}^{K}\frac{N_k}{N}T_{i,k}(x)$, where $W_d(x)$ and $\sum_{k=1}^{K}\frac{N_k}{N}T_{i,k}(x)$ denote the CDF-s for the duration of a white hole in the case where the white begins with one channel and ends with another one, and where the white hole begins and ends with the same channel, respectively. The probability of the latter scenario is

$$P_{lw} = \sum_{k=1}^{K} \frac{Q_{tot}\,(N_k/N)}{\int_{x=0}^{\infty}(\int_{y=0}^{x} t_{i,k}(y)dy)a_k(x)dx} \tag{21}$$



(a) Mean value. (b) Coefficient of variation. (c) Skewness.

Fig. 7. White hole duration statistics as functions of number of channels and activity factor of the channel. Exponential distribution of channel active and idle times.

where $Q_{tot} = \prod_{k=1}^{K} \left(\int_{x=0}^{\infty} (\int_{y=0}^{x} t_{i,k}(y)dy) a_k(x)dx\right)^{N_k}$.

From the probability density function of complete channel availability $w(x) = \frac{dW(x)}{dx}$, we can derive its LST as

$$W^*(s) = \int_{x=0}^{\infty} e^{-sx} w(x)dx \tag{22}$$

and the first three moments as

$$\overline{W} = -\frac{dW^s(s)}{ds}\bigg|_{s=0} \tag{23}$$

$$var(W) = \frac{d^2W^s(s)}{ds^2}\bigg|_{s=0} - \overline{W}^2 \tag{24}$$

$$S_w = \frac{(-1)^3 \left.\frac{d^3W^*(s)}{ds^3}\right|_{s=0} - 3 \left.\frac{d^2W^*(s)}{ds^2}\right|_{s=0} \overline{W} + 2\overline{W}^3}{var(W)^{3/2}} \tag{25}$$

Quantitatively, white holes can be described with the diagrams shown in Fig. 7 for exponential distribution and cycle time $T_{cyc} = 3000$ unit time slots. Since both idle and active channel times follow exponential distribution, behavior of white holes is virtually identical to the behavior of black holes, except that p_{on} values from one case correspond to $p_{off} = 1 - p_{on}$ from the other.

Analogous to the approach outlined in section III-B, duration of white hole can also be approximated with an exponential distribution with the rate of $\lambda_w = \sum_{k=1}^{K} N_k a_k(0)$.

We note that these results can be very useful in the analysis of sequence-based protocols for rendezvous in cognitive networks [5], [11]. These protocols require that all channels are available throughout the duration of the rendezvous sequence; a collision with primary user transmission means that the sequence needs to be restarted, possibly after recalculation and shortening. The probability that a rendezvous sequence will be able to finish without a collision can be calculated on the basis of sequence duration as well as on the duration of a white hole. The existence of a simple exponential approximation means that the results of such analysis will apply regardless of the actual distribution of channel active and idle times.

V. Modeling gray holes

It is also important to evaluate the impact of gray holes. However, due to tractability problems we will assume that there is only one type of primary users with exponentially distributed active and idle times, and simplify the equations by omitting the index of primary user type in analysis. Probability that $m < N$ channels are simultaneously busy is

$$P_{a,m} = \binom{N}{m} p_{on}^m (1 - p_{on})^{N-m} \tag{26}$$

CDF for the duration of a m-channel gray hole has to be derived in two scenarios similar to the back and white holes, using the probability that the active time is shorter than residual active time on any channel, which may be calculated as

$$P_{lm} = \left(\int_{x=0}^{\infty} (\int_{y=0}^{x} t_a(y)dy) b(x)dx\right)^{m-1} \tag{27}$$

Therefore, CDF for the duration of a gray hole $D_m(x)$ can be calculated as

$$D_{d,m}(x) = 1 - (1 - B(x))^m \tag{28}$$

$$D_{s,m}(x) = T_a(x) \tag{29}$$

$$D_m(x) = (1 - P_{lm}) D_{d,m}(x) + P_{lm} D_{s,m}(x) \tag{30}$$

The pdf of the duration of a gray hole is $d_m(x) = \frac{dD_m(x)}{dx}$, and its LST is

$$D^*_{d,m}(s) = \int_{x=0}^{\infty} e^{-sx} d_m(x)dx \tag{31}$$

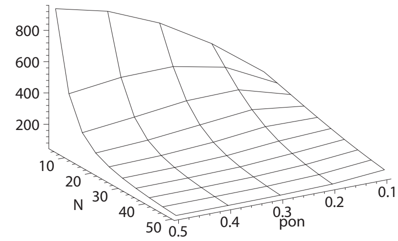
Mean duration and variance of a m-channel gray hole is

$$\overline{D_m} = \int_{x=0}^{\infty} x d_m(x)dx \tag{32}$$

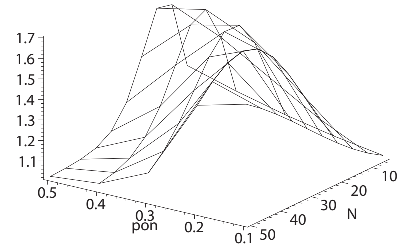
$$var(D_m) = \int_{x=0}^{\infty} (x - \overline{D_m})^2 d_m(x)dx \tag{33}$$

Since the probability that a gray hole with N channels (i.e., a black home) occurs is $P_r = 1 - p_{off}^N$, we can calculate pdf for the duration of any gray hole containing m channels or more (up to N, in which case it is, in fact, a black hole) as

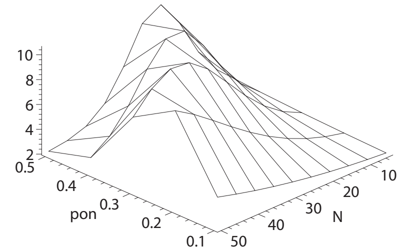
$$d_{m-N}(x) = \sum_{i=m}^{N} P_{a,i} \frac{d_i(x)}{P_r} \tag{34}$$



(a) Mean value at $T_{cyc} = 3000$ unit time slots.

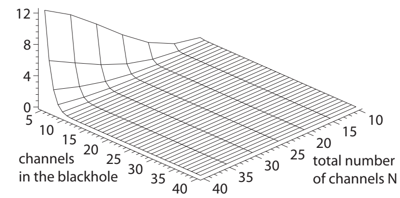


(b) Coefficient of variation at $T_{cyc} = 3000$ unit time slots.

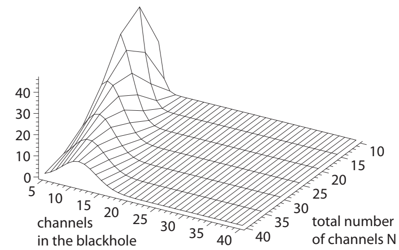


(c) Skewness at $T_{cyc} = 3000$ unit time slots.

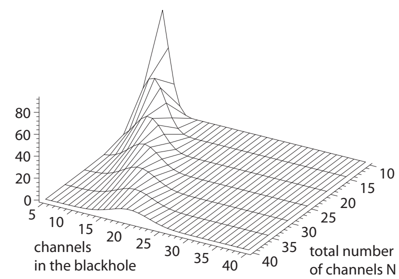
Fig. 8. Gray hole statistic descriptors under exponential distribution of channel active and idle times.



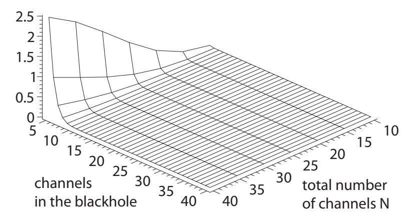
(a) Impact of gray hole at primary user activity factor of $p_{on} = 0.1$.



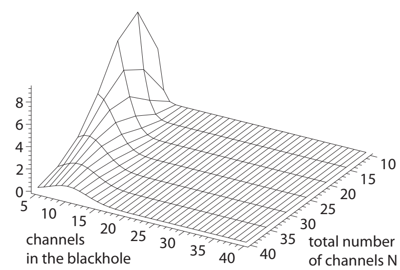
(b) Impact of gray hole at primary user activity factor of $p_{on} = 0.3$.



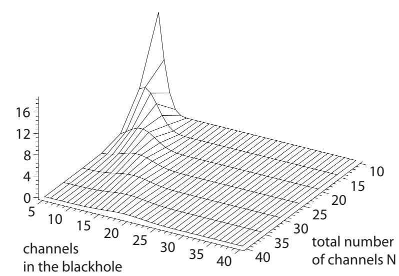
(c) Impact of gray hole at primary user activity factor of $p_{on} = 0.5$.



(d) Impact of one channel in a gray hole, at primary user activity factor of $p_{on} = 0.1$.



(e) Impact of one channel in a gray hole, at primary user activity factor of $p_{on} = 0.3$.



(f) Impact of one channel in a gray hole, at primary user activity factor of $p_{on} = 0.5$.

Fig. 9. Impact of gray holes under exponentially distributed active and idle times of primary users with mean cycle time $\overline{T_{cyc}} = 3000$). Top row: impact calculated per (40), bottom row: impact calculated per (40) and normalized by the number of channels in a gray hole.

Therefore the LST for the duration of such a gray hole can be calculated as

$$T_r^*(s) = \sum_{i=m}^{N} P_{a,i} D_{d,i}^*(s)/P_r \tag{35}$$

while its first three moments are

$$\overline{T_r} = -\left.\frac{dT_r^*(s)}{ds}\right|_{s=0} \tag{36}$$

$$var(T_r) = \left.\frac{d^2T_r^*(s)}{ds^2}\right|_{s=0} - \overline{T_r}^2 \tag{37}$$

$$S_r = (-1)^3 \left.\frac{d^3T_r^*(s)}{ds^3}\right|_{s=0} / var(T_r)^{3/2} \tag{38}$$

using the probability that the duration of holes of size m and larger does not exceed a given value S, calculated as

$$P_s = \int_{x=0}^{S} d_{m-N}(x)dx \tag{39}$$

In this manner, we may obtain the number of channels required to maintain P_s below some specified threshold, which should provide useful guidance in the design of cognitive secondary networks.

Fig. 8 presents the distribution of gray hole duration, as calculated per (35), under exponentially distributed active and idle times of primary users with mean cycle time of $T_{cyc} = 3000$ unit slots. Mean gray hole duration is noticeably longer, in some cases more than twice, than the duration of a black hole under the same parameter values; this is to be expected, as a black hole is just a gray hole of maximum size N. It is interesting to note that mean value of gray hole duration exhibits similar but not quite the same shape of decrease as that of black holes with the increase in the number of channels and the decrease of primary user activity factor.

Coefficient of variation and skewness exhibit peaks which are slightly larger than their black hole counterparts; at the same time, they are also dependent on the primary user activity factor, unlike their black hole counterparts.

It is also of interest to evaluate the impact of individual gray holes. We may define the impact of an m-channel gray hole as a product of its duration and frequency of occurrence:

$$Ins_m = P_{a,m}\overline{D_m} \tag{40}$$

The diagrams in the top row of Fig. 9 presents weighted duration of the black hole according to (40). The diagrams in the bottom row show those same values but further normalized with the number of channels in the gray hole in question. We notice that largest impact is brought by the gray holes of size close to $p_{on}N$. However, with the increase of activity factor of the primary user, a larger number of gray holes and their respective channels begin to exhibit significant higher impact.

Both black and gray holes cause collisions that disrupt the operation of cognitive networks. (In fact, the cognitive network is unable to operate throughout the duration of a black hole.) The analysis presented above offers important quantitative insight into the behavior of such holes; these values can then be used to design advanced MAC protocols for cognitive networks that will minimize the disruption caused by black and gray holes. For example, results in Section III-C can be used to determine parameter values for a recovery algorithm, similar to that described in [17], that will probabilistically guarantee network recovery in case of a collision.

VI. Conclusion

In this paper we have examined the behavior of spectral holes and analyzed the probability distribution of their duration. Our results provide quantitative insight into the impact of black holes that interrupt the operation of a cognitive network. Overall, when the number of channels used by the cognitive network is small – say, less than about 10 – the distribution of black hole duration depends very much on the distribution of primary user active and idle times. Moreover, this distribution exhibits a hyper-exponential distribution with a long right tail, since the coefficient of variation is above 1 and the skewness coefficient is above 2.

However, in systems with a larger number of channels the duration of black holes follows a probability distribution that is very nearly exponential; in addition, the time interval between successive black holes is also shown to follow a nearly exponential distribution. Our analysis shows that both of these effects are virtually independent of the actual distribution of active and idle times on different channels. These results can be used to aid in performance analysis of MAC-level algorithms and protocols for cognitive networks.

We have also examined the behavior of white holes, which is analogous to that of black holes, as well as the behavior of gray holes, the impact of which has been quantified using a simple proxy measure.

Our future work will focus on developing MAC protocols for cognitive networks that offer increased resilience to black and gray holes, as well as protocols for quick network formation and recovery.

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