cps720: Probability Formula Sheet

1 Conditional Probability

- $P(E|F)$ is the probability of the occurence of event E given that F occured and is given as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

  Knowing that F occured reduces the sample space to F, and the part of it where E also occured is $E \cap F$. Note that this formula is well defined only if $P(F) > 0$.

- Because $\cap$ is commutative, we have

$$P(E \cap F) = P(E|F)P(F) = P(F|E)P(E)$$

- When random events F_i are mutually exclusive and exhaustive set of events (ie, events F_i and F_j cannot co-occur if $i \neq j$ and $\bigcup_i F_i$ is the whole sample space), then

$$P(E) = \sum_i P(E \cap F_i) = \sum_i P(E|F_i)P(F_i)$$

- If random events E and F are independent, we have $P(E|F) = P(E)$ and thus

$$P(E \cap F) = P(E)P(F)$$

  That is, knowledge that whether F has occured does not change the probability that E occurs.

2 Probability Distribution and Density Function

- The probability distribution function $F(.)$ of a random variable X for any real number a is

$$F(a) = P\{X \leqslant a\}$$

  and we have

$$P\{a < X \leqslant b\} = F(b) - F(a)$$

- If X is a discreet random variable then

$$F(a) = \sum_{\forall x \leqslant a} P(x)$$

  where $P(.)$ is the function defined as $P(a) = P\{X = a\}$, and $P\{X = a\}$ means the probability of the event that a random variable X has the value a in a random experiment.

3 Conditional Expectation

- Expectation, or expected value, or mean of a discrete random variable X, denoted by $E[X]$, is the sum of the values that X can take multiplied by the probabilities that each value can occur

$$E[X] = \sum_i x_i P(x_i)$$

  where x_i is one of the possible values of X and $P(X_i) = P\{X = x_i\}$ is the probability of the event that the variable X takes the value x_i in a random experiment. In other words, it is a weighted average where each value is weighted by the probability that X takes that value.

- It is a well-known result in probability theory that, if a and b are real numbers, and X and Y are random variables, then

$$E[aX + b] = aE[X] + b \tag{1}$$
$$E[X + Y] = E[X] + E[Y] \tag{2}$$

- The conditional expectation of a discrete random variable X given that a ramdom variable Y takes the value y, i.e. $Y = y$, is defined by

$$E[X|Y = y] = \sum_i x_i P\{X = x_i | Y = y\}$$

- Define a random variable $E[X|Y]$ that takes values $E[X|Y = y_j]$ with probability $P\{Y = y_j\}$. Because the conditional expectation $E[X|Y]$ is a random variable, if Y is a discrete random variable, then by the definition of expectation we can compute the expected value of $E[X|Y]$ as

$$E[E[X|Y]] = \sum_j E[X|Y = y_j] \cdot P\{Y = y_j\}.$$

  It's also a standard result that

$$E[X] = E[E[X|Y]] = \sum_j E[X|Y = y_j] P\{Y = y_j\}$$

  This equation states that to calculate $E[X]$, we can take a weighted average of the conditional expected value of X given that $Y = y_j$, each of the terms $E[X|Y = y_j]$ being weighted by the probability $P\{Y = y_j\}$ of the event on which it is conditioned.

4 Weak Law of Large Numbers

Let $X = \{X_t\}_{t=1}^N$ be a set of N independent and identically distributed (iid) random variables each having mean μ and a finite variance σ^2. Then for any $\epsilon > 0$

$$P\left\{ \left| \frac{\sum_t X_t}{N} - \mu \right| > \epsilon \right\} \to 0 \qquad as \quad N \to \infty$$

That is, the average value of X over N trials converges to the mean value of X as N increases.

5 Special Distributions

5.1 Uniform Distribution

- A random variable X is uniformly distributed over the interval [a, b] if its density function is given by

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

- If X is uniform, its expected value and variance are

$$E[X] = \frac{a+b}{2}, \qquad Var(X) = \frac{(b-a)^2}{12}$$

5.2 Normal (Gaussian) Distribution

- A random variable X is normally distributed with mean μ and variance σ^2, denoted as $N(\mu, \sigma^2)$, if its density function is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \quad -\infty < x < \infty$$

6 Generating Random Numbers in C

- The following C library functions can be used to generate a random value according to a uniform distribution between 0 and 1:

```
#include <stdio.h>
int rand(void);
void srand(unsigned int seed);
```

  Do a 'man rand' on the command line to see more details about these functions

- To generate a random number according to the Gausian distribution, the GNU Scientific Library can be used. For information about how to use this library, about Unix random generators, or about the Gaussian random functions, check out these links

```
http://www.gnu.org/software/gsl/manual/gsl-ref_2.html
http://www.gnu.org/software/gsl/manual/gsl-ref_17.html#SEC272
http://www.gnu.org/software/gsl/manual/gsl-ref_19.html#SEC288
```

7 Generating Random Numbers in Java

Information related to random number generating in Java can be found at

```
http://java.sun.com/j2se/1.5.0/docs/api/java/util/Random.html
```