Relations
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1 Relations (10.1)

Definition 1

1. A relation from a set \(A\) to a set \(B\) is a subset \(R\) of \(A \times B\).

2. Given \((x, y) \in R\) we say that \(x\) is related to \(y\) and write \(xRy\).

3. If \((x, y) \notin R\) we say that \(x\) is not related to \(y\) and write \(x \not\in R\).

4. If \(A = B\), we say that \(R\) is a binary relation on \(A\).

Note: Order matters. If \(xRy\) this does not necessarily mean that \(yRx\).

Example 2
Let \(A = \{0, 1, 2\}, B = \{3, 4, 5\}\). \(R = \{(0, 3), (0, 4), (1, 3)\}\).
So \(0R3, 0R4, 1R3\).
But \(3 \not\in R\), \(4 \not\in R\), \(3 \not\in R\).
0 is related to 2 different elements of \(B\).
There are 2 elements of \(A\) (0 and 1) which are related to \(3 \in B\).
2 \(\in A\) is related to no elements in \(B\).
There are no elements in \(A\) related to \(5 \in B\).

We may represent a relation by a diagram in which a line is drawn between two elements if they are related.

If \(A = B\), i.e. \(R\) is a binary relation on \(A\), we need only draw \(A\), we can then connect points of \(A\) to each other as needed.

Example 3
\(A = \{0, 1, 2\}, R = \{(0, 0), (0, 1, 1, 2)\}\)
Example 4

1. > on \( \mathbb{N} \), \( A = B = \mathbb{N} \),
   \[ \forall x, y \in \mathbb{N}, xRy \iff x > y. \]
   So \( 1 > 0 \Rightarrow 1R0 \), but \( 0 \not\sim 1 \).
   \[ R = \{(1,0),(2,0),(2,1),(3,0),(3,1),(3,2),\ldots\}. \]

2. Let \( P = \{x \mid x \text{ is a person }\}. \)
   Consider the relation ‘Child of’ on \( P \), given by
   \[ \forall x, y \in P, xRy \iff x \text{ is } y \text{’s child}. \]

3. = on \( Q \).
   \[ A = B = Q, \forall x, y \in Q, xRy \iff x = y. \]
   We will use the following definition of equality in \( Q \):
   Given \( \frac{a}{b}, \frac{c}{d} \in Q \), we say that \( \frac{a}{b} = \frac{c}{d} \) if and only if
   \[ \frac{a}{\gcd(a,b)} = \frac{c}{\gcd(c,d)} \text{ and } \frac{b}{\gcd(a,b)} = \frac{d}{\gcd(c,d)}. \]
   So, \( \frac{1}{2} = \frac{1}{2}, \frac{1}{2} = \frac{2}{4}, \frac{2}{4} = \frac{1}{2}, \frac{1}{2} = \frac{3}{6}, \frac{2}{4} = \frac{3}{6}, \ldots \)

4. Congruence modulo \( n \ (\equiv_n) \).
   \[ A = B = \mathbb{Z}, \forall x, y \in \mathbb{Z}, xRy \iff x \mod n = y \mod n. \]
   We write \( x \equiv y \pmod{n} \).
   So, for example, if \( n = 6 \),
   \[ 1 \equiv 1 \pmod{6}, 1 \equiv 7 \pmod{6}, 1 \equiv 13 \pmod{6}, 1 \equiv 19 \pmod{6}, \ldots \]
   \[ 7 \equiv 1 \pmod{6}, 7 \equiv 7 \pmod{6}, 7 \equiv 13 \pmod{6}, 7 \equiv 19 \pmod{6}, \ldots \]
   \[ 2 \equiv 2 \pmod{6}, 2 \equiv 8 \pmod{6}, 2 \equiv 14 \pmod{6}, 2 \equiv 20 \pmod{6}, \ldots \]
   \[ 3 \equiv 3 \pmod{6}, 3 \equiv 9 \pmod{6}, 3 \equiv 15 \pmod{6}, 3 \equiv 21 \pmod{6}, \ldots \]
   \[ 4 \equiv 4 \pmod{6}, 4 \equiv 10 \pmod{6}, 4 \equiv 16 \pmod{6}, 4 \equiv 22 \pmod{6}, \ldots \]
   \[ 5 \equiv 5 \pmod{6}, 5 \equiv 11 \pmod{6}, 5 \equiv 17 \pmod{6}, 5 \equiv 23 \pmod{6}, \ldots \]
   \[ 0 \equiv 0 \pmod{6}, 0 \equiv 6 \pmod{6}, 0 \equiv 12 \pmod{6}, 0 \equiv 18 \pmod{6}, \ldots \]

Definition 5 Given a binary relation \( R \) from \( A \) to \( B \) the \textbf{inverse relation}, denoted \( R^{-1} \), is given by \( R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\} \).

Example 6

1. Let \( A = \{0, 1, 2\}, B = \{3, 4, 5\} \). \( R = \{(0, 3), (0, 4), (1, 3)\} \).
   Then \( R^{-1} = \{(3, 0), (4, 0), (3, 1)\} \).
2. Inverse of $>$ on $\mathbb{N}$.
\[ \forall a, b \in \mathbb{N}, \ aRb \iff a > b \iff b < a. \]
So $>^{-1}$ is $<$, i.e. $(a, b) \in > \Rightarrow (b, a) \in <$.

3. (Child of)$^{-1}$.
\[ \forall x, y \in P, x \text{ is the child of } y \text{ if and only if } y \text{ is the parent of } x. \]
So (Child of)$^{-1}$ is (Parent of).

4. $=$ on $\mathbb{Q}$.
\[ \text{Note that by the definition of equality in } \mathbb{Q} \text{ for all } x, y \in \mathbb{Q}, x = y \iff y = x. \]
So $=^{-1}$ is $=$.

5. Congruence modulo $n$ ($\equiv_n$).
\[ a \equiv b \pmod{n} \]
\[ \iff a \mod{n} = b \mod{n} \]
\[ \iff b \mod{n} = a \mod{n} \]
\[ \iff b \equiv a \pmod{n}. \]
So $\equiv^{-1}$ is $\equiv$.

The last two examples are self inverse. We showed that $aRb \iff bRa$.

\textbf{Definition 7} Given $n \in \mathbb{Z}^+$ and sets $A_1, A_2, \ldots, A_n$, an $n$-ary relation on $A_1, A_2, \ldots, A_n$ is a subset $R$ of $A_1 \times A_2 \times \ldots \times A_n$. 
If $n = 2$ it is called a binary relation.
If $n = 3$ it is called a ternary relation.
If $n = 4$ it is called a quaternary relation.

\textbf{Example 8}

1. Every subroutine (function) in a computer program is a relation between its inputs and its output(s).
\begin{verbatim}
int f(int a, char *b, int c) {
    ... return (int) d;
}
\end{verbatim}
This is a quaternary relation, relating the inputs $a, b$ and $c$ with the output $d$.
i.e. $(a, b, c, d) \in R$ if and only if $f$ returns $d$ on the inputs $a, b$ and $c$.
If $f$ returns a set of values (an array), rather than a single value, we say that $(a, b, c, d) \in R$
for each $d$ in the array.
In this case $A_1 = A_3 = A_4 = \mathbb{Z}$, and $A_2 = \Sigma_A^*$, where $\Sigma_A$ is the ASCII character set.
2. Consider a database, each record of which contains 4 fields.
Each record in the database looks like \((x, y, z, w)\),
where \(x \in A_1\), \(y \in A_2\), \(z \in A_3\), \(w \in A_4\), for some sets \(A_1, A_2, A_3, A_4\).
We define a quaternary relation \(R\) by \((x, y, z, w) \in R\) if and only if \((x, y, z, w)\) is a record in the database.
This is called a relational database.

2  Equivalence Relations

2.1 Reflexive, Symmetric and Transitive Relations (10.2)

There are three important properties which a relation may, or may not, have.

**Definition 9** Given a binary relation, \(R\), on a set \(A\):

1. \(R\) is called Reflexive if \(\forall x \in A, xRx\).

2. \(R\) is called Symmetric if \(\forall x, y \in A, xRy \Rightarrow yRx\).

3. \(R\) is called Transitive if \(\forall x, y, z \in A, (xRy \land yRz) \Rightarrow xRz\).

4. A relation which is reflexive, symmetric and transitive is called an equivalence relation.

**Example 10**

1. \(A = \{0, 1, 2\}\), \(R = \{(0, 0), (1, 1), (1, 2), (2, 1), (0, 2), (2, 0)\}\)

   \(2R2\) so not reflexive.

   For each \(x, y \in A\) \(xRy \Rightarrow yRx\) (by exhaustion). So \(R\) is symmetric

   \(1R2\) and \(2R0\), but \(1R0\), so \(R\) is not transitive.

   So \(R\) is not an equivalence relation (neither reflexive nor transitive).

2. \(A = \{0, 1, 2\}\), \(R = \{(0, 0), (1, 1), (2, 2), (1, 2), (2, 1)\}\)

   For each \(x, y \in A\), \(xRx\), so \(R\) is reflexive.

   For each \(x, y \in A\), \(xRy \Rightarrow yRx\) (by exhaustion), so \(R\) is symmetric.

   By reflexivity, \(\forall x \in A\), \(xRx \land xRx \Rightarrow xRx\).

   \((1R1 \land 1R2) \Rightarrow 1R2\), \((1R2 \land 2R1) \Rightarrow 1R1\),

   \((1R2 \land 2R2) \Rightarrow 1R2\), \((2R1 \land 1R1) \Rightarrow 2R1\).

   So by exhaustion \(R\) is transitive.

   Thus \(R\) is an equivalence relation.

3. \(>\) on \(\mathbb{N}\)

   \(0 \not> 0\), so \(>\) is not reflexive.

   \(1 > 0\), but \(0 \not> 1\), so \(>\) is not symmetric.

   Let \(a, b, c \in \mathbb{N}\) with \(a > b\) and \(b > c\), then \(a > c\), so \(>\) is transitive.

   Thus \(>\) is not an equivalence relation (not reflexive or symmetric).
4. (Child of) on \( P \).

No person is their own child, so not reflexive.
If \( x \) is \( y \)'s child, then \( y \) is not \( x \)'s child, so not symmetric.
If \( x \) is the child of \( y \) and \( y \) is the child of \( z \), then \( x \) is not the child of \( z \), so not transitive.
So Child of is not an equivalence relation.

5. \( = \) on \( Q \)

**Reflexivity:** Let \( \frac{a}{b} \in Q, \frac{a}{b} = \frac{a}{b} \). So \( = \) is reflexive.

**Symmetry:** Let \( \frac{a}{b}, \frac{c}{d} \in Q, \) with \( \frac{a}{b} = \frac{c}{d} \).
By the definition of \( = \) in \( Q, \frac{c}{d} = \frac{a}{b} \). So \( = \) is symmetric.

**Transitivity** Let \( \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in Q \) with \( \frac{a}{b} = \frac{c}{d} \) and \( \frac{c}{d} = \frac{e}{f} \).
Then \( \frac{c}{d} = \frac{gcd(c, d)}{gcd(a, b)} \) and \( \frac{e}{f} = \frac{gcd(e, f)}{gcd(c, d)} \), so \( \frac{a}{b} = \frac{gcd(a, b)}{gcd(e, f)} \). (Algebra)
Also \( \frac{a}{b} = \frac{gcd(a, b)}{gcd(e, f)} \) and \( \frac{e}{f} = \frac{gcd(e, f)}{gcd(c, d)} \), so \( \frac{a}{b} = \frac{gcd(a, b)}{gcd(e, f)} \). (Algebra)

Thus \( \frac{a}{b} = \frac{e}{f} \), and so \( = \) is transitive. \( \square \)

Thus \( = \) is an equivalence relation.

6. Congruence modulo \( n (\equiv_n) \).

**Reflexivity:** Let \( a \in \mathbb{Z}, a \mod n = a \mod n \), so \( a \equiv a \pmod{n} \).
So \( \equiv_n \) is reflexive.

**Symmetry:** Let \( a, b \in \mathbb{Z}, \) with \( a \equiv b \pmod{n} \).
Thus \( a \mod n = b \mod n \), so \( b \equiv a \pmod{n} \).
So \( \equiv_n \) is symmetric.

**Transitivity:** Let \( a, b, c \in \mathbb{Z}, \) with \( a \equiv b \pmod{n} \) and \( b \equiv c \pmod{n} \).
Then \( a \mod n = b \mod n \) and \( b \mod n = c \mod n \),
so \( a \mod n = c \mod n \), i.e. \( a \equiv c \pmod{n} \).
So \( \equiv_n \) is transitive.
Thus \( \equiv_n \) is an equivalence relation.

### 2.2 Transitive Closure

Suppose we have a binary relation \( R \) on a set \( A \) which is not transitive.
This means that there are triples of elements \( a, b, c \in A \) with \( aRb \) and \( bRc \), but \( aRc \).
Suppose that we create a new relation, \( R' \), by adding \( (a, c) \) to the relation for each such triple.
The resulting relation, \( R' \), will be transitive.
\( R' \) is called the transitive closure of \( R \).

**Definition 11** Given a set \( A \) and a binary relation \( R \) on \( A \), the **transitive closure** of \( R \) is the relation \( R' \) which satisfies the following properties:
1. $R'$ is transitive.
2. $R \subseteq R'$.
3. If $S$ is any other transitive relation on $A$ which contains $R$, then $R' \subseteq S$.

Example 12 $A = \{0, 1, 2\}$, $R = \{(0, 0), (0, 1), (1, 2)\}$

3 Equivalence Classes

Definition 13 Given an equivalence relation $R$ on a set $A$, for each $a \in A$ we define the equivalence class of $a$, denoted $[a]$, to be the set $[a] = \{x \in A \mid xRa\}$.

Note $R$ is reflexive so $\forall a \in A$, $aRa$, i.e. $a \in [a]$.

Lemma 14 (10.3.2) Given an equivalence relation $R$ on a set $A$, $\forall a, b \in A$, $aRb \iff [a] = [b]$.

S.W.P. (See Epp p. 563)

Lemma 15 (10.3.3) Given an equivalence relation $R$ on a set $A$, $\forall a, b \in A$, $a \not R b \iff [a] \cap [b] = \emptyset$.

S.W.P. (See Epp p. 564)

Theorem 16 Given an equivalence relation $R$ on a set $A$, the distinct equivalence classes of $R$ are a partition of $A$.

Examples

1. $= \text{on } \mathbb{Q}$.

   Equivalence classes are $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ such that $\frac{a}{b} = \frac{c}{d}$. Thus

   $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\}$,

   $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\}$,

   $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\}$, etc.

   Note that $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \ldots, \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \ldots, \text{ etc.}$
When we talk about a number in \( \mathbb{Q} \) we usually don’t care what form it is in, \( \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots \), we consider these all equivalent. Thus our usual conception of \( \mathbb{Q} \) is in fact an idea of the equivalence classes of \( \mathbb{Q} \) under \( = \). We should really write \( \left[ \frac{1}{2} \right] \) rather than \( \frac{1}{2} \).

This illustrates the usefulness of equivalence classes, they ‘take out’ extra information from the set, leaving us with only that information which we consider useful for a particular purpose.

2. Congruence modulo \( n \) (\( \equiv \)).

The equivalence classes are \( a, b \in \mathbb{Z} \) such that \( a \mod n = b \mod n \).

\[
\begin{align*}
[0] &= \{a \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \text{ such that } a = kn\} \\
[1] &= \{a \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \text{ such that } a = kn + 1\} \\
[2] &= \{a \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \text{ such that } a = kn + 2\} \\
&\vdots \\
[n-2] &= \{a \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \text{ such that } a = kn - 2\} \\
[n-1] &= \{a \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \text{ such that } a = kn - 1\}
\end{align*}
\]

Note that \( \ldots = [-2n] = [-n] = [0] = [n] = [2n] = \ldots \), \( \ldots = [-2n + 1] = [-n + 1] = [1] = [n + 1] = [2n + 1] = \ldots \), etc.

3. Consider the relation \( S \) on \( \mathbb{R}^2 \), given by

\( \forall \ u, v \in \mathbb{R}^2, \ uSv \iff \exists a \in \mathbb{R}^+ \text{ such that } u = av \).

\( S \) is an equivalence relation (Exercise: prove this).

The equivalence classes of \( S \) are all those vectors in \( \mathbb{R}^2 \) which have the same direction.

Each unit vector defines an equivalence class.

Thus the equivalence classes ‘take out’ information about magnitude.

We may represent each equivalence class by a point on the unit circle.

4 Antisymmetry and Order Relations

**Definition 17** A relation \( R \) on a set \( A \) is called antisymmetric if and only if

\[ \forall a, b \in A, aRb \land bRa \Rightarrow a = b \]

**Notes**

1. In general if a relation has any element which is related to a different element then if it is antisymmetric then it is not symmetric and visa versa.

2. Antisymmetry allows for the possibility that \( R \) is reflexive.

**Example 18**

1. \( A = \{0, 1, 2\}, \ R = \{(0, 0), (1, 1), (2, 2)\} \)

   This relation is both symmetric and antisymmetric. It is also reflexive.
2. \( A = \{0, 1, 2\} \),
   
   (a) \( R_a = \{(0, 0), (0, 1), (1, 2), (1, 0)\} \).
   This relation is not antisymmetric (\((0, 1) \in R \) and \((1, 0) \in R_a\)), nor reflexive (\((2, 2) \notin R_a\)),
   nor transitive (\((0, 2) \notin R_a\), but \((0, 1) \in R \) and \((1, 2) \in R\)).
   
   (b) \( R_b = \{(0, 0), (0, 1), (1, 2)\} \).
   This relation is antisymmetric, but neither reflexive (\((2, 2) \notin R_b\)), nor transitive (\((0, 2) \notin R_b\)).
   
   (c) \( R_c = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 2)\} \).
   This relation is antisymmetric and reflexive, but not transitive (\((0, 2) \notin R_c\)).
   
   (d) \( R_d = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 2), (0, 2)\} \).
   This relation is antisymmetric, it is also transitive and reflexive, it is the transitive closure
   of \( R_c \) above.

3. \( > \) on \( \mathbb{N} \).

   There are no \( a, b \in \mathbb{N} \) such that \( a > b \) and \( b > a \), so the predicate of the implication is always
   false so and this relation is antisymmetric. \( \forall a, b \in \mathbb{N} \).

4. The Divides Relation on \( \mathbb{Z}^+ \).

   \[ \forall a, b \in \mathbb{Z}^+, aRb \iff a \mid b \]

   \( \mid \) is antisymmetric: \( a \mid b \) and \( b \mid a \) implies that \( a = b \).

   It is also reflexive (\( a \mid a \) for all \( a \in \mathbb{Z}^+ \)) and transitive (transitivity of divisibility).

5. The Subset relation.

   Let \( A \) be any set, consider the set of subsets of \( A \), \( \mathcal{P}(A) \).

   \[ \forall B, C \in \mathcal{P}(A), BRC \iff B \subseteq C \]

   eg \( A = \{0, 1, 2\} \)
   
   \( \{0\} \subseteq \{0, 1\} \), so \( \{0\}R\{0, 1\} \). \( \{0, 1\} \subseteq \{0, 1, 2\} \), so \( \{0, 1\}R\{0, 1, 2\} \) etc.
   
   But \( \{2\} \not\subseteq \{0, 1\} \), so \( \{2\} \not\in \mathcal{P}\{0, 1\} \).

   This relation is reflexive (definition of set equality) and transitive (transitivity of \( \subseteq \) (5.2.1
   \#3)).

**Definition 19** Given a binary relation, \( R \), on a set \( A \) it is called a partial order if it is reflexive,
antisymmetric and transitive.

The idea is that in an antisymmetric relation, given two unequal elements \( a, b \in A \) then either \( aRb \)
or \( bRa \), but not both. If \( aRb \) then \( a \preceq b \), but if \( bRa \) then \( b \preceq a \). We use the symbol \( \preceq \) to avoid
confusion with \( \leq \). This defines an order on the elements of \( A \).

However, not all elements are necessarily related, hence the term partial order.
Definition 20 Given a partial order $R$ on a set $A$.

1. For any pair of elements $a, b \in A$ they are called comparable if either $aRb$ or $bRa$; otherwise they are called noncomparable.

2. If $R$ is a partial order and every pair of elements are comparable, then $R$ is called a total order.

3. A subset $C \subseteq A$ is a chain if every pair of elements of $C$ are comparable.

4. The length of the chain is one less than the number of elements of the chain.

The idea of a partial order is that if $aRb$, then $a \leq b$. Since it is antisymmetric there is no ambiguity. However, not all elements are comparable, hence the term partial order. The chains define lines of comparable elements.

Notes

- A partial order restricted to a chain is a total order.
- In a total order the entire set $A$ is a chain.

Example 21

1. $\leq$ on $\mathbb{N}$ is a total ordering.

2. The Divides Relation on $\mathbb{Z}^+$ is a partial ordering. It is not a total ordering.
   
   $2 \leq 4 \leq 8 \leq 16 \ldots$, so powers of 2 form a chain.

   Note that $4 | 12$, but $8 \nmid 12$, so 8 and 12 are not comparable and $\{4, 8, 12\}$ is not a chain.

   Also $3 \nmid 4$ so 3 and 4 are not comparable.

3. The Subset relation on $A = \{0, 1, 2\}$.
   
   $\{\{0\}, \{0, 1\}, \{0, 1, 2\}\}$ and $\{\{1\}, \{0, 1\}, \{0, 1, 2\}\}$ are both chains, but $\{\{0\}, \{1\}, \{0, 1\}, \{0, 1, 2\}\}$ is not, since $\{0\}$ and $\{1\}$ are not comparable.

Definition 22 Given a partial order $R$ on a set $A$.

- An element $a \in A$ is called maximal if $\forall x \in A$, either $x \leq a$ or $x$ and $a$ are not comparable.

- An element $a \in A$ is called a greatest element (or a maximum) if $\forall x \in A, x \leq a$.

- An element $a \in A$ is called minimal if $\forall x \in A, aRx$ or $x$ and $a$ are not comparable.

- An element $a \in A$ is called a least element (or minimum) if $\forall x \in A, x \leq a$. 