Relations

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1 Relations (10.1)

Definition 1

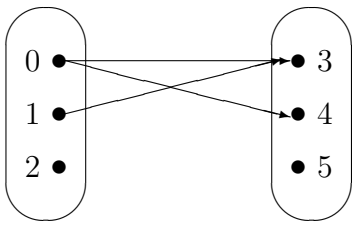
1. *A* <u>*relation*</u> *from a set A to a set B is a subset R of $A \times B$.*
2. *Given $(x, y) \in R$ we say that x is* <u>*related*</u> *to y and write xRy.*
3. *If $(x, y) \notin R$ we say that x is* <u>*not related*</u> *to y and write $x \not R y$.*
4. *If $A = B$, we say that R is a* <u>*binary relation on A*</u>.

Note: Order matters. If xRy this does **not** necessarily mean that yRx.

Example 2

Let $A = \{0, 1, 2\}$, $B = \{3, 4, 5\}$. $R = \{(0, 3), (0, 4), (1, 3)\}$.
So $0R3$, $0R4$, $1R3$.
But $3 \not R 0$, $4 \not R 0$, $3 \not R 1$.
0 is related to 2 different elements of B.
There are 2 elements of A (0 and 1) which are related to $3 \in B$.
$2 \in A$ is related to no elements in B.
There are no elements in A related to $5 \in B$.

We may represent a relation by a diagram in which a line is drawn between two elements if they are related.



If $A = B$, i.e. R is a binary relation on A, we need only draw A, we can then connect points of A to each other as needed.

Example 3

$A = \{0, 1, 2\}$, $R = \{(0, 0), (0, 1), (1, 2)\}$

Example 4

1. $>$ on $\mathbb{N}$, $A = B = \mathbb{N}$,

   $\forall x, y \in \mathbb{N},\ xRy \Leftrightarrow x > y$.

   So $1 > 0 \Rightarrow 1R0$, but $0 \not R 1$ since $0 \not> 1$.

   $R = \{(1,0), (2,0), (2,1), (3,0), (3,1), (3,2), \ldots\}$.

2. Let $P = \{x \mid x \text{ is a person }\}$.

   Consider the relation 'Child of' on P, given by
   $\forall\, x, y \in P,\ xRy$ if and only if x is y's child.

3. $=$ on $\mathbb{Q}$.

   $A = B = \mathbb{Q}$, $\forall\, x, y \in \mathbb{Q}, xRy \Leftrightarrow x = y$.

   We wil use the following definition of equality in $\mathbb{Q}$:
   Given $\dfrac{a}{b}, \dfrac{c}{d} \in \mathbb{Q}$, we say that $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if

   $$\frac{a}{\gcd(a,b)} = \frac{c}{\gcd(c,d)} \quad \text{and} \quad \frac{b}{\gcd(a,b)} = \frac{d}{\gcd(c,d)}.$$

   So, $\dfrac{1}{2} = \dfrac{1}{2}$, $\dfrac{1}{2} = \dfrac{2}{4}$, $\dfrac{2}{4} = \dfrac{1}{2}$, $\dfrac{1}{2} = \dfrac{3}{6}$, $\dfrac{2}{4} = \dfrac{3}{6}$, $\ldots$

4. Congruence modulo n ($\equiv_n$).

   $A = B = \mathbb{Z}$, $\forall\, x, y \in \mathbb{Z},\ xRy \Leftrightarrow x \bmod n = y \bmod n$.

   We write $x \equiv y \pmod{n}$.

   So, for example, if $n = 6$,

   $1 \equiv 1 \pmod 6,\ 1 \equiv 7 \pmod 6,\ 1 \equiv 13 \pmod 6,\ 1 \equiv 19 \pmod 6,\ \ldots$

   $7 \equiv 1 \pmod 6,\ 7 \equiv 7 \pmod 6,\ 7 \equiv 13 \pmod 6,\ 7 \equiv 19 \pmod 6,\ \ldots$

   $2 \equiv 2 \pmod 6,\ 2 \equiv 8 \pmod 6,\ 2 \equiv 14 \pmod 6,\ 2 \equiv 20 \pmod 6,\ \ldots$

   $3 \equiv 3 \pmod 6,\ 3 \equiv 9 \pmod 6,\ 3 \equiv 15 \pmod 6,\ 3 \equiv 21 \pmod 6,\ \ldots$

   $4 \equiv 4 \pmod 6,\ 4 \equiv 10 \pmod 6,\ 4 \equiv 16 \pmod 6,\ 4 \equiv 22 \pmod 6,\ \ldots$

   $5 \equiv 5 \pmod 6,\ 5 \equiv 11 \pmod 6,\ 5 \equiv 17 \pmod 6,\ 5 \equiv 23 \pmod 6,\ \ldots$

   $0 \equiv 0 \pmod 6,\ 0 \equiv 6 \pmod 6,\ 0 \equiv 12 \pmod 6,\ 0 \equiv 18 \pmod 6,\ \ldots$

Definition 5 *Given a binary relation R from A to B the* <u>*inverse relation*</u>*, denoted R^{-1}, is given by $R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\}$.*

Example 6

1. Let $A = \{0,1,2\}$, $B = \{3,4,5\}$. $R = \{(0,3), (0,4), (1,3)\}$.

   Then $R^{-1} = \{(3,0), (4,0), (3,1)\}$.

2. Inverse of $>$ on $\mathbb{N}$.

   $\forall a, b \in \mathbb{N},\ aRb \Leftrightarrow a > b \Leftrightarrow b < a$.

   So $>^{-1}$ is $<$, i.e. $(a, b) \in > \Rightarrow\ (b, a) \in <$

3. (Child of)$^{-1}$.

   $\forall x, y \in P,\ x$ is the child of y if and only if y is the parent of x.

   So (Child of)$^{-1}$ is (Parent of).

4. $=^{-1}$ on $\mathbb{Q}$.

   Note that by the definition of equality in $\mathbb{Q}$ for all $x, y \in \mathbb{Q},\ x = y \Leftrightarrow y = x$.

   So $=^{-1}$ is $=$.

5. Congruence modulo n ($\equiv_n$).

   $a \equiv b \pmod{n}$
   $\Leftrightarrow a \bmod n = b \bmod n$
   $\Leftrightarrow b \bmod n = a \bmod n$
   $\Leftrightarrow b \equiv a \pmod{n}$.

   So $\equiv^{-1}$ is $\equiv$

The last two examples are <u>self inverse</u>. We showed that $aRb \Leftrightarrow bRa$.

Definition 7 *Given $n \in \mathbb{Z}^+$ and sets $A_1, A_2, \ldots, A_n$, an <u>n-ary relation on $A_1, A_2, \ldots, A_n$</u> is a subset R of $A_1 \times A_2 \times \ldots \times A_n$.*
If $n = 2$ it is called a binary relation.
If $n = 3$ it is called a ternary relation.
If $n = 4$ it is called a quaternary relation.

Example 8

1. Every subroutine (function) in a computer program is a relation between its inputs and its output(s).

   int f(int a, char *b, int c) {
   ⋮
   return (int) d;
   }

   This is a quaternary relation, relating the inputs a, b and c with the output d.
   i.e. $(a, b, c, d) \in R$ if and only if f returns d on the inputs a, b and c.

   If f returns a set of values (an array), rather than a single value, we say that $(a, b, c, d) \in R$ for each d in the array.

   In this case $A_1 = A_3 = A_4 = \mathbb{Z}$, and $A_2 = \Sigma_A^*$, where Σ_A is the ASCII character set.

2. Consider a database, each record of which contains 4 fields.
   Each record in the database looks like (x, y, z, w),
   where $x \in A_1,\ y \in A_2,\ z \in A_3,\ w \in A_4$, for some sets A_1, A_2, A_3, A_4.

   We define a quaternary relation R by $(x, y, z, w) \in R$ if and only if (x, y, z, w) is a record in the database.

   This is called a relational database.

2 Equivalence Relations

2.1 Reflexive, Symmetric and Transitive Relations (10.2)

There are three important properties which a relation may, or may not, have.

Definition 9 *Given a binary relation, R, on a set A:*

1. *R is called <u>Reflexive</u> if $\forall\, x \in A,\ xRx$.*
2. *R is called <u>Symmetric</u> if $\forall\, x, y \in A,\ xRy \Rightarrow yRx$.*
3. *R is called <u>Transitive</u> if $\forall\, x, y, z \in A,\ (xRy \wedge yRz) \Rightarrow xRz$.*
4. *A relation which is reflexive, symmetric and transitive is called an <u>equivalence relation</u>.*

Example 10

1. $A = \{0, 1, 2\}$, $R = \{(0, 0), (1, 1), (1, 2), (2, 1), (0, 2), (2, 0)\}$

   $2 \not R 2$ so not reflexive.
   For each $x, y \in A$ $xRy \Rightarrow yRx$ (by exhaustion). So R is symmetric
   $1R2$ and $2R0$, but $1 \not R 0$, so R is not transitive.

   So R is not an equivalence relation (neither reflexive nor transitive).

2. $A = \{0, 1, 2\},\ R = \{(0, 0), (1, 1), (2, 2), (1, 2), (2, 1)\}$

   For each $x, y \in A$, xRx, so R is reflexive.
   For each $x, y \in A$, $xRy \Rightarrow yRx$ (by exhaustion), so R is symmetric.
   By reflexivity, $\forall x \in A$, $(xRx \wedge xRx) \Rightarrow xRx$.
   $(1R1 \wedge 1R2) \Rightarrow 1R2$, $\quad (1R2 \wedge 2R1) \Rightarrow 1R1$,
   $(1R2 \wedge 2R2) \Rightarrow 1R2$, $\quad (2R1 \wedge 1R1) \Rightarrow 2R1$.
   So by exhaustion R is transitive.

   Thus R is an equivalence relation.

3. $>$ on $\mathbb{N}$

   $0 \not> 0$, so $>$ is not reflexive.
   $1 > 0$, but $0 \not> 1$, so $>$ is not symmetric.
   Let $a, b, c \in \mathbb{N}$ with $a > b$ and $b > c$, then $a > c$, so $>$ is transitive.

   Thus $>$ is not an equivalence relation (not reflexive or symmetric).

4. (Child of) on P.

   No person is their own child, so not reflexive.
   If x is y's child, then y is not x's child, so not symmetric.
   If x is the child of y and y is the child of z, then x is not the child of z, so not transitive.

   So Child of is not an equivalence relation.

5. $=$ on $\mathbb{Q}$

   Reflexivity: Let $\frac{a}{b} \in \mathbb{Q}$, $\frac{a}{b} = \frac{a}{b}$. So $=$ is reflexive.

   Symmetry: Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, with $\frac{a}{b} = \frac{c}{d}$.
   By the definition of $=$ in $\mathbb{Q}$, $\frac{c}{d} = \frac{a}{b}$. So $=$ is symmetric.

   Transitivity Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$ with $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$.
   Then $\frac{c}{\gcd(c,d)} = \frac{a}{\gcd(a,b)}$ and $\frac{c}{\gcd(c,d)} = \frac{e}{\gcd(e,f)}$, so $\frac{e}{\gcd(e,f)} = \frac{a}{\gcd(a,b)}$. (Algebra)
   Also $\frac{d}{\gcd(c,d)} = \frac{b}{\gcd(a,b)}$ and $\frac{d}{\gcd(c,d)} = \frac{f}{\gcd(e,f)}$, so $\frac{b}{\gcd(a,b)} = \frac{f}{\gcd(e,f)}$. (Algebra)
   Thus $\frac{a}{b} = \frac{e}{f}$, and so $=$ is transitive. $\square$

   Thus $=$ is an equivalence relation.

6. Congruence modulo n ($\equiv_n$).

   Reflexivity: Let $a \in \mathbb{Z}$, $a \bmod n = a \bmod n$, so $a \equiv a \pmod{n}$.
   So $\equiv_n$ is reflexive.

   Symmetry: Let $a, b \in \mathbb{Z}$, with $a \equiv b \pmod{n}$.
   Thus $a \bmod n = b \bmod n$, so $b \equiv a \pmod{n}$.
   So $\equiv_n$ is symmetric.

   Transitivity: Let $a, b, c \in \mathbb{Z}$, with $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$.
   Then $a \bmod n = b \bmod n$ and $b \bmod n = c \bmod n$,
   so $a \bmod n = c \bmod n$, i.e. $a \equiv c \pmod{n}$.
   So $\equiv_n$ is transitive.

   Thus $\equiv_n$ is an equivalence relation.

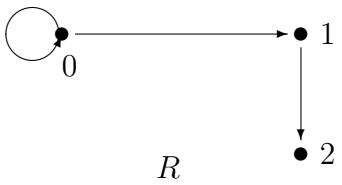
2.2 Transitive Closure

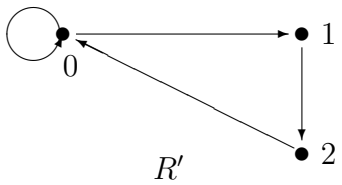
Suppose we have a binary relation R on a set A which is not transitive.
This means that there are triples of elements $a, b, c \in A$ with aRb and bRc, but $a \not R c$.
Suppose that we create a new relation, R', by adding (a, c) to the relation for each such triple.
The resulting relation, R', will be transitive.
R' is called the <u>transitive closure</u> of R.

Definition 11 *Given a set A and a binary relation R on A, the <u>transitive closure</u> of R is the relation R' which satisfies the following properties:*

1. *R' is transitive.*
2. *$R \subseteq R'$.*
3. *If S is any other transitive relation on A which contains R, then $R' \subseteq S$.*

Example 12 $A = \{0, 1, 2\}$, $R = \{(0,0), (0,1), (1,2)\}$





3 Equivalence Classes

Definition 13 *Given an equivalence relation R on a set A, for each $a \in A$ we define* <u>*the equivalence class*</u> *of a, denoted $[a]$, to be the set $[a] = \{x \in A \mid xRa\}$.*

Note R is reflexive so $\forall\, a \in A$, aRa, i.e. $a \in [a]$.

Lemma 14 (10.3.2) *Given an equivalence relation R on a set A, $\forall a, b \in A$, $aRb \Leftrightarrow [a] = [b]$.*

S.W.P. (See Epp p. 563)

Lemma 15 (10.3.3) *Given an equivalence relation R on a set A, $\forall a, b \in A$, $a \not R b \Leftrightarrow [a] \cap [b] = \phi$.*

S.W.P. (See Epp p. 564)

Theorem 16 *Given an equivalence relation R on a set A, the distinct equivalence classes of R are a partition of A.*

Examples

1. $=$ on $\mathbb{Q}$.

   Equivalence classes are $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ such that $\frac{a}{b} = \frac{c}{d}$. Thus

   $\left[\frac{1}{2}\right] = \left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\right\}$,

   $\left[\frac{1}{3}\right] = \left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots\right\}$,

   $\left[\frac{2}{3}\right] = \left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\right\}$, etc.

   Note that $\left[\frac{1}{2}\right] = \left[\frac{2}{4}\right] = \left[\frac{3}{6}\right] = \ldots$, $\quad \left[\frac{1}{3}\right] = \left[\frac{2}{6}\right] = \left[\frac{3}{9}\right] = \ldots$, $\quad$ etc.

When we talk about a *number* in $\mathbb{Q}$ we usually don't care what form it is in, $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$, we consider these all equivalent. Thus our usual conception of $\mathbb{Q}$ is in fact an idea of the equivalence classes of $\mathbb{Q}$ under $=$. We should really write $\left[\frac{1}{2}\right]$ rather than $\frac{1}{2}$.
This illustrates the usfuleness of equivalence classes, they 'take out' extra information from the set, leaving us with only that information wich we consider useful for a particular purpose.

2. Congruence modulo n ($\equiv_n$).

   The equivalence classes are $a, b \in \mathbb{Z}$ such that $a \bmod n = b \bmod n$.

   $[0] = \{a \in \mathbb{Z} \mid \exists\, k \in \mathbb{Z} \text{ such that } a = kn\}$
   $[1] = \{a \in \mathbb{Z} \mid \exists\, k \in \mathbb{Z} \text{ such that } a = kn + 1\}$
   $[2] = \{a \in \mathbb{Z} \mid \exists\, k \in \mathbb{Z} \text{ such that } a = kn + 2\}$
   $\vdots$
   $[n-2] = \{a \in \mathbb{Z} \mid \exists\, k \in \mathbb{Z} \text{ such that } a = kn - 2\}$
   $[n-1] = \{a \in \mathbb{Z} \mid \exists\, k \in \mathbb{Z} \text{ such that } a = kn - 1\}$

   Note that $\ldots = [-2n] = [-n] = [0] = [n] = [2n] = \ldots$,
   $\ldots = [-2n+1] = [-n+1] = [1] = [n+1] = [2n+1] = \ldots$, etc.

3. Consider the relation S on $\mathbb{R}^2$, given by
   $\forall\, \mathbf{u}, \mathbf{v} \in \mathbb{R}^2,\ \mathbf{u}S\mathbf{v} \Leftrightarrow \exists\, a \in \mathbb{R}^+$ such that $\mathbf{u} = a\mathbf{v}$.

   S is an equivalence relation (Exercise: prove this).

   The equivalence classes of S are all those vectors in $\mathbb{R}^2$ which have the same direction.

   Each unit vector defines an equivalence class.

   Thus the equivalence classes 'take out' information about magnitude.

   We may represent each equivalence class by a point on the unit circle.

4 Antisymmetry and Order Relations

Definition 17 *A relation R on a set A is called* <u>*antisymmetric*</u> *if and only if*

$$\forall a, b \in A, aRb \wedge bRa \Rightarrow a = b$$

Notes

1. In general if a relation has any element which is related to a different element then if it is *antisymmetric* then it is **not** *symmetric* and visa versa.

2. Antisymmetry allows for the possibility that R is reflexive.

Example 18

1. $A = \{0, 1, 2\},\ R = \{(0,0), (1,1), (2,2)\}$

   This relation is both symmetric and antisymmetric. It is also reflexive.

2. $A = \{0, 1, 2\}$,

   (a) $R_a = \{(0, 0), (0, 1), (1, 2), (1, 0)\}$.
   This relation is not antisymmetric ($(0, 1) \in R$ and $(1, 0) \in R_a$), nor reflexive ($(2, 2) \notin R_a$), nor transitive ($(0, 2) \notin R_a$, but $(0, 1) \in R$ and $(1, 2) \in R$).

   (b) $R_b = \{(0, 0), (0, 1), (1, 2)\}$.
   This relation is antisymmetric, but neither reflexive ($(2, 2) \notin R_b$), nor transitive ($(0, 2) \notin R_b$).

   (c) $R_c = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 2)\}$.
   This relation is antisymmetric and reflexive, but not transitive ($(0, 2) \notin R_c$).

   (d) $R_d = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 2), (0, 2)\}$
   This relation is antisymmetric, it is also transitive and reflexive, it is the transitive closure of R_c above.

3. $>$ on $\mathbb{N}$.

   There are no $a, b \in \mathbb{N}$ such that $a > b$ and $b > a$, so the predicate of the implication is always false so and this releation is antisymmetric. $\forall a, b \in \mathbb{N}$.

4. The Divides Relation on $\mathbb{Z}^+$.

$$\forall a, b \in \mathbb{Z}^+, aRb \Leftrightarrow a \mid b$$

   $\mid$ is antisymmetric: $a \mid b$ and $b \mid a$ implies that $a = b$.

   It is also reflexive ($a \mid a$ for all $a \in \mathbb{Z}^+$) and transitive (transitivity of divisibility).

5. The Subset relation.

   Let A be any set, consider the set of subsets of A, $\mathcal{P}(A)$.

$$\forall B, C \in \mathcal{P}(A), BRC \Leftrightarrow B \subseteq C$$

   eg $A = \{0, 1, 2\}$
   $\{0\} \subseteq \{0, 1\}$, so $\{0\}R\{0, 1\}$. $\{0, 1\} \subseteq \{0, 1, 2\}$, so $\{0, 1\}R\{0, 1, 2\}$ etc.
   But $\{2\} \not\subseteq \{0, 1\}$, so $\{2\} \not R \{0, 1\}$.

   This relation is reflexive (definition of set equality) and transitive (transitivity of $\subseteq$ (5.2.1 #3)).

Definition 19 *Given a binary relation, R, on a set A it is called a* <u>*partial order*</u> *if it is reflexive, antisymmetric and transitive.*

The idea is that in an antisymmetric relation, given two unequal elements $a, b \in A$ then either aRb or bRa, but not both. If aRb then $a \preceq b$, but if bRa then $b \preceq a$. We use the symbol $\preceq$ to avoid confusion with $\leq$. This defines an order on the elements of A.
However, not all elements are necessarily related, hence the term *partial* order

Definition 20 *Given a partial order R on a set A.*

1. *For any pair of elements $a, b \in A$ they are called* <u>*comparable*</u> *if either aRb or bRa; otherwise they are called* <u>*noncomparable.*</u>
2. *If R is a partial order and every pair of elemets are comparable, then R is called a* <u>*total order*</u>*.*
3. *A subset $C \subseteq A$ is a* <u>*chain*</u> *if every pair of elements of C are comparable.*
4. *The* <u>*length of the chain*</u> *is one less than the number of elements of the chain.*

The idea of a partial order is that if aRb, then $a \preceq b$. Since it is antisymmetric there is no ambiguity. However, not all elements are comparable, hence the term *partial* order.
The chains define lines of comparable elements.

Notes

- A partial order restricted to a chain is a total order.
- In a total order the entire set A is a chain.

Example 21

1. $\leq$ on $\mathbb{N}$ is a total ordering.
2. The Divides Relation on $\mathbb{Z}^+$ is a partial ordering. It is not a total ordering.

   $2 \preceq 4 \preceq 8 \preceq 16 \ldots$, so powers of 2 form a chain.

   Note that $4 \mid 12$, but $8 \not| \, 12$, so 8 and 12 are not comparable and $\{4, 8, 12\}$ is not a chain.

   Also $3 \not| \, 4$ so 3 and 4 are not comparable.
3. The Subset relation on $A = \{0, 1, 2\}$.

   $\{\{0\}, \{0, 1\}, \{0, 1, 2\}\}$ and $\{\{1\}, \{0, 1\}, \{0, 1, 2\}\}$ are both chains, but $\{\{0\}, \{1\}, \{0, 1\}, \{0, 1, 2\}\}$ is not, since $\{0\}$ and $\{1\}$ are not comparable.

Definition 22 *Given a partial order R on a set A.*

- *An element $a \in A$ is called* <u>*maximal*</u> *if $\forall x \in A$, either $x \preceq a$ or x and a are not comparable.*
- *An element $a \in A$ is called a* <u>*greatest element*</u> *(or a* maximum*) if $\forall x \in A, x \preceq a$.*
- *An element $a \in A$ is called* <u>*minimal*</u> *if $\forall x \in A, aRx$ or x and a are not comparable.*
- *An element $a \in A$ is called a* <u>*least element*</u> *(or* minimum*) if $\forall x \in A, x \preceq a$.*