Notes on Game Theory
MTH 503

Two person zero-sum games

Two players, **R** and **C** (row and column players).

R can make moves $R_1, R_2, \cdots R_m$
C can make moves $C_1, C_2, \cdots C_n$.

The pay-off to **R** for choosing move R_i and being countered by **C**'s move C_j is $p(R_i, C_j)$.

P is the pay-off matrix (to **R**); $[\mathbf{P}]_{ij} = p(R_i, C_j) = p_{ij}$

Zero-sum game: The pay-off to **R** = loss to **C**, i.e., the pay-off to **C** for choosing move C_j after **R** chooses R_i is $-p(R_i, C_j)$

As a matrix:

	C_1	.	.	.	C_n
R_1	p_{11}	p_{12}	.	.	p_{1n}
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
R_m	p_{m1}	.	.	.	p_{mn}

Suppose the game is played many times and on average, **R** chooses R_i with frequency w_i and **C** chooses C_j with frequency x_j $(i = 1, \ldots, m, \quad j = 1, \ldots, n)$:

$$w_i = \lim_{k\to\infty} \frac{\#\text{ times } \mathbf{R} \text{ chooses } R_i \text{ in } k \text{ games}}{k} \qquad x_j = \lim_{k\to\infty} \frac{\#\text{ times } \mathbf{C} \text{ chooses } C_j \text{ in } k \text{ games}}{k}$$

So $w_1 + \cdots + w_m = 1$ with all $w_i \geq 0$ and $x_1 + \cdots + x_n = 1$ with all $x_j \geq 0$.

A strategy for **R** is such a vector $\mathbf{w} = w_1, \ldots, w_m$, and a strategy for **C** is a such vector $\mathbf{x} = x_1, \ldots, x_n$.

Goal of each player: To determine a strategy that guarantees a minimum expected payoff (in the long run) *regardless* of the moves of the other player. Such a strategy is called an optimal strategy and will be denoted by $\mathbf{w}^*$ and $\mathbf{x}^*$ for **R** and **C** respectively.

Are there optimal strategies for **R** and **C**? If there is, then **R** will realize his minimum expected pay-off and **C** will realize her expected minimum pay-off. The value v of the game is the minimum expected pay-off to **R**: $v = \sum_{i,j} w_i^* x_j^* p_{ij} \; = \; \mathbf{w}^*\mathbf{P}(\mathbf{x}^*)^t$.

Pure Strategies: **R** and **C** will always choose the same move, i.e.,

$$\begin{aligned} w_i &= 1 \quad \text{for some } i \text{ and } w_l = 0 \text{ for all } l \neq i, \\ x_j &= 1 \quad \text{for some } j \text{ and } x_k = 0 \text{ for all } k \neq j \end{aligned}$$

Mixed Strategies: **R** and **C** will choose different moves; all $w_i < 1$ and all $x_j < 1$.

Example 1

	C_1	C_2		min	max
R_1	2	-2	$\rightarrow$	-2	
					-2
R_2	-3	4	$\rightarrow$	-3	
	$\downarrow$	$\downarrow$			
max	2	4			
min	2				

So **R**'s min-max strategy is R_1; if **R** chooses R_1 then no matter what moves **C** makes, **R** will get at least -2 pay-off (i.e., a minimum loss). The value of the min-max strategy is -2. While **C**'s max-min strategy is C_1; if **C** chooses C_1, then no matter what moves **R** makes, **C** won't lose more than 2. The value of the max-min strategy is 2.

Here, the value of the min-max strategy $\neq$ the value of the max-min strategy. When this happens the min-max and max-min strategies are not compatible; **R** and **C** cannot both persue these pure strategies. If **R** sticks to R_1, then **C** will (or should!) eventually choose C_2 to maximize her pay-off. But then **R** will (or should!) switch to R_2 to maximize his pay-off, etc. So **R** and **C** will be forced to choose mixed strategies. However, if the value of min-max strategy = the value of the max-min strategy (as in Example 2), then the max-min and min-max strategies are compatible. When this happens we say that the game has a saddle point.

Example 2

	C_1	C_2		min	max
R_1	0	5	$\rightarrow$	0	
					2
R_2	2	4	$\rightarrow$	2	
	$\downarrow$	$\downarrow$			
max	2	5			
min	2				

Here, R_2 is the min-max strategy and C_1 is the max-min strategy. **R** can always choose R_2 while **C** can always choose C_1. The value of this game is $v = [0,1]\begin{bmatrix} 0 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2$. **R** will obtain a pay-off of 2 if he sticks to R_2 and if **C** sticks to C_1. However, if **C** deviates from C_1 then **R** will do <u>better</u> than 2. Similarly, **C** will pay a maximum of 2 if she sticks to C_1 and if **R** sticks to R_2. However, if **R** deviates from R_2 then **C** will pay <u>less</u> than 2. Thus, these strategies are <u>optimal</u>. (Remark: A game can have more than one saddle point.)

Back to Example 1. <u>**R**'s goal</u>: Suppose **R** chooses R_1 and R_2 with frequencies w_1 and w_2. Then

pay-off (to **R**) when **C** chooses C_1:	$2w_1 - 3w_2$
pay-off (to **R**) when **C** chooses C_2:	$-2w_1 + 4w_2$

Let u = average pay-off to **R** (i.e., in the long run). Then, to play it <u>safe</u>, **R** wants to choose w_1 and w_2 so that he receives a guaranteed expected pay-off *no matter what* **C** *does*. Thus, **R** wants to maximize u subject to

$$\begin{array}{rcrcl} 2w_1 & - & 3w_2 & \geq & u \\ -2w_1 & + & 4w_2 & \geq & u \\ w_1 & + & w_2 & = & 1 \\ w_1, & & w_2 & \geq 0 & \\ & & u & \in R & \end{array} \tag{1}$$

Solution: $\mathbf{w}^* = (.6364,\ .3636),\ u^* = .1818$.

<u>**C**'s goal</u>: Suppose **C** chooses C_1 and C_2 with frequencies x_1 and x_2. Then

pay-off (to **R**!) when **R** chooses R_1:	$2x_1 - 2x_2$
pay-off (to **R**!) when **R** chooses R_2:	$-3x_1 + 4x_2$

Let z = average pay-off to **R** (!). Then **C** wants to choose x_1 and x_2 so that she receives a guaranteed expected pay-off *no matter what* **R** *does* (the <u>safe</u> choice). Thus, **C** wants to minimize y subject to

$$\begin{array}{rcrcl} 2x_1 & - & 2x_2 & \leq & y \\ -3x_1 & + & 4x_2 & \leq & y \\ x_1 & + & x_2 & = & 1 \\ x_1, & & x_2 & \geq 0 & \\ & & y & \in R & \end{array} \tag{2}$$

Solution: $\mathbf{x}^* = (.5454,\ .4545),\ y^* = .1818$.

<u>Homework</u>: Show that the LP problem (2) is the dual of the LP problem (1).

<u>Result</u>: The optimal solutions $\mathbf{w}^*$ and $\mathbf{x}^*$ of (1) and (2) are the optimal strategies for **R** and **C**. Furthermore, $\mathbf{w}^*$ and $\mathbf{x}^*$ are <u>compatible</u>. The <u>value of the mixed strategies</u> is $v = \mathbf{w}^*\mathbf{P}(\mathbf{x}^*)^t = .1818$.

Fundamental Theorem of Games: In the absence of saddle points, optimal mixed strategies always exist.

Extensions:

- Two-person nonconstant-sum games
- N-person games
- Differential games

Applications:

- Economics
- Control theory

References

Bazaraa et. al., *Linear Programing and Network Flows*, Exercise 6.25, page 306.

A. Rapoport, *Two-Person Game Theory.* (Dover paperback.)

W. Winston, *Operations Research*, Chapter 15.